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SOUND AND MATHEMATICS

A Project
Presented to the
Faculty of
California State University,
San Bernardino

In Partial Fulfillment
of the Requirements for the Degree
Master of Arts
in
Mathematics

by
Nancy Jean Parham

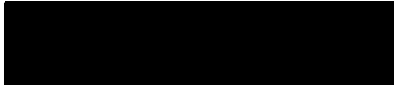
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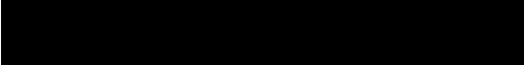
by
Nancy Jean Parham
June 1992


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ABSTRACT

This paper investigates the connection between sound and mathematics. In particular, we study the vibrations of a plucked string, such as a guitar string, and of a hollow cylinder fixed at both ends set in motion by striking its surface. The investigation found the connection between sound and mathematics to be the eigenvalue of the Laplacian differential operator of the vibrating object which corresponds to a particular solution to the wave equation. The wave equation is a second order linear differential equation $\frac{\partial^2 \Phi}{\partial t^2} = c^2 \Delta \Phi$.

Using the eigenvalue λ , the frequency is calculated by $f = \sqrt{\lambda} / 2\pi$. The frequencies produced determine the notes on a musical scale.

This project is dedicated to
my children Devon and Dustin,
and my loving husband Rick who took
a week off from work to do the laundry.

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INTRODUCTION

This paper will investigate the connection between musical sound and mathematics. I first became interested in the connection between sound and mathematics last summer while making a set of wind chimes. The chimes were made of brass tubing, and the different lengths produced different sounds as the tubes were struck. My investigation first led to the case of the vibrating string. From properties of physics I was able to derive the wave equation. The wave equation is an equation in which many models of physical phenomena follow. Any natural occurring property that involves periodic motion, vibration and oscillation are properties that follow the wave equation. Some example of waves are; the guitar string fixed at both ends and then plucked, the hollow cylinder's surface struck with an object, seismic waves propagating beneath the surface of the earth during an earthquake, or shock waves from a sonic boom produced from the space shuttle entering the earth's atmosphere. This paper will consider models of the first two situations, the fixed string and the cylinder. In the case of the fixed string I will examine two cases, that case of both ends fixed and then with only one end fixed. In the case of the cylinder, I will consider both ends fixed.

Solutions of these three cases require the use of partial differential equations. From the solution to the partial differential equations, we can determine the frequencies of the vibrating object and thus calculate the notes from the

musical scale that are produced.

Note: "model" means the mathematical approximation to the physical system.

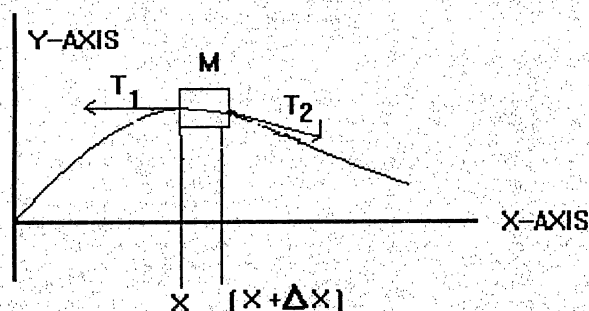
§1.1 DERIVE THE WAVE EQUATION FOR THE STRING

When a string is plucked, the string vibrates and produces sound we perceive as a musical tone. The study of physical phenomena in which waves are produced involves the wave equation. The wave equation is a second partial differential equation $\frac{\partial^2 \Phi}{\partial t^2} = c^2 \frac{\partial^2 \Phi}{\partial x^2}$. [7] More generally,

$$\frac{\partial^2 \Phi}{\partial t^2} = c^2 \Delta \Phi \quad \text{where } \Delta \text{ is the Laplacian. In } \mathbb{R}^1, \quad \Delta = \frac{\partial^2}{\partial x^2},$$

and in \mathbb{R}^n $\Delta = \sum \frac{\partial^2}{\partial x_i^2}$ The following is a derivation of

the wave equation from physical properties of the string.



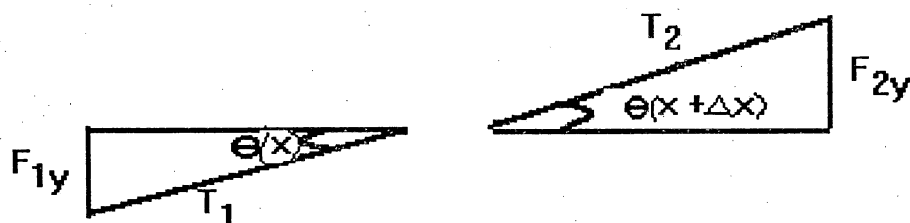
The diagram depicts a string as it is set in motion. The tension in the string is represented by T_1 and T_2 , where T_1 is the force in the negative x direction, and T_2 is the force in the positive x direction. The mass M is the mass of the piece of string between x and $x + \Delta x$. We assume that Δx is very small.

Recall that force is equal to mass times acceleration,

where the acceleration is the second partial derivative of y with respect to t , $a = \frac{\partial^2 y}{\partial t^2}$. So $F = ma$ is the net force.

Forces in the x direction cancel each other out since they are the same force in opposite directions. This leaves the remaining forces of T_1 and T_2 . If $F = ma$, then $F = m \frac{\partial^2 y}{\partial t^2}$

In the triangle, the opposite side of angle $\theta(x)$



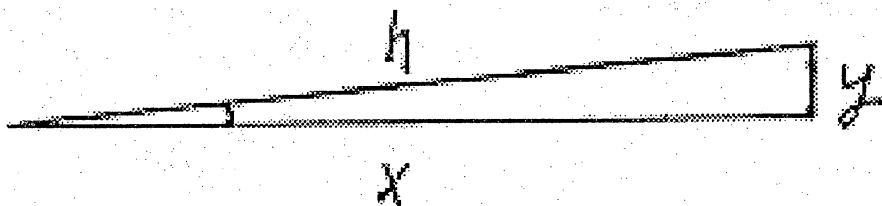
These triangles show the angles made by T_1 and T_2 in the figure on page three.

$$F_{1y} = T_1 \sin \theta(x)$$

Similarly, considering the opposite side of angle $\theta(x + \Delta x)$, we have $F_{2y} = T_2 \sin \theta(x + \Delta x)$.

Since the wave is rising, F_{2y} is greater than F_{1y} . The net force is $F_{2y} - F_{1y}$. So $F_{2y} - F_{1y} = T_2 \sin \theta(x + \Delta x) - T_1 \sin \theta(x)$. The displacement of a plucked string is small, so the vibrating string remains almost horizontal, and the angles θ

are quite small. When the θ is very small, the tangent of θ is approximately equal to the sine of θ . (Estimate from the Maclaurin series)



$$\begin{aligned} \text{So } F_{2y} - F_{1y} &= T_2 \sin \theta(x + \Delta x) - T_1 \sin \theta(x) \\ &\approx T_2 \tan \theta(x + \Delta x) - T_1 \tan \theta(x) \end{aligned}$$

But $T_1 = T_2$, since the tension in the string is the same at T_1 as it is for T_2 . Therefore;

$$F_{2y} - F_{1y} = T(\tan \theta(x + \Delta x) - \tan \theta(x)).$$

Recall that $\tan \theta(x)$ is the slope of the tangent lines at x and $x + \Delta x$, or the partial derivative $\frac{\partial y}{\partial x}$.

$$\text{So the net force is } F_{2y} - F_{1y} = T \left(\frac{\partial y}{\partial x} \Big|_{\text{at } x + \Delta x} - \frac{\partial y}{\partial x} \Big|_{\text{at } x} \right)$$

But the net force is also $F = ma$, which is $m \frac{\partial^2 y}{\partial t^2}$;

$$\text{therefore } F_{2y} - F_{1y} = m \frac{\partial^2 y}{\partial t^2},$$

which implies $m \frac{\partial^2 y}{\partial t^2} \approx T \left(\frac{\partial y}{\partial x} \Big|_{\text{at } x+\Delta x} - \frac{\partial y}{\partial x} \Big|_{\text{at } x} \right)$

Define u to equal the linear mass density, $u = \frac{m}{l}$, where m

is the mass of the whole string, and l is the length of the string. So mass of a small piece of string is equal to $u(\Delta x)$, where (Δx) is the length of a small piece of the string.

Therefore

$$u(\Delta x) \frac{\partial^2 y}{\partial t^2} \approx T \left(\frac{\partial y}{\partial x} \Big|_{\text{at } x+\Delta x} - \frac{\partial y}{\partial x} \Big|_{\text{at } x} \right)$$

So
$$\frac{\frac{\partial^2 y}{\partial t^2}}{\frac{\partial y}{\partial x}} \approx \frac{T \left(\frac{\partial y}{\partial x} \Big|_{\text{at } x+\Delta x} - \frac{\partial y}{\partial x} \Big|_{\text{at } x} \right)}{(\Delta x)}$$

Observe that in the limit we have the x derivative of $\frac{\partial y}{\partial x}$,

since $\lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ is the derivative of the function

$f = \frac{\partial y}{\partial x}$, which is the second partial of y with respect to x .

We therefore have the wave equation where T/u is a positive constant

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{u} \frac{\partial^2 y}{\partial x^2}$$

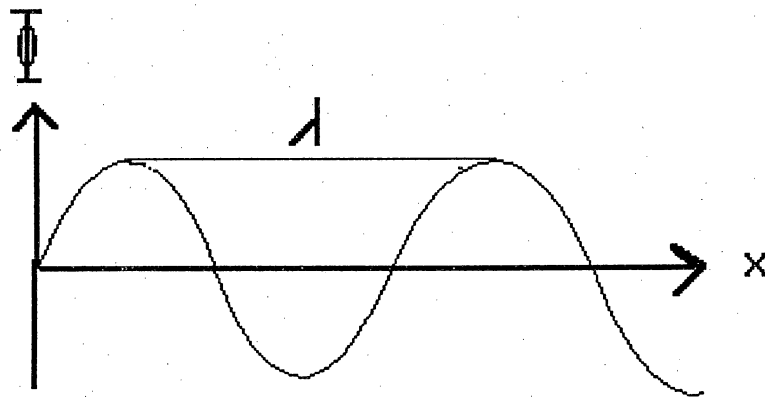
We have just derived the wave equation using physical properties. The wave equation has a constant T/u , which we will now interpret. If we have a periodic wave $\phi(x,t)$, the

constant T/u becomes c^2 , where c is the speed of the wave. To prove $T/u = c^2$, we need some definitions .

Definition: The frequency f , is equal to the number of vibrations per second. $f = \frac{\text{vib}}{\text{sec}}$, where the units of the

frequency are measured in hertz, Hz. The period P , is the number of seconds per vibration, or $P = \frac{1}{f}$. The wave

length λ is the length of one complete cycle of the wave.



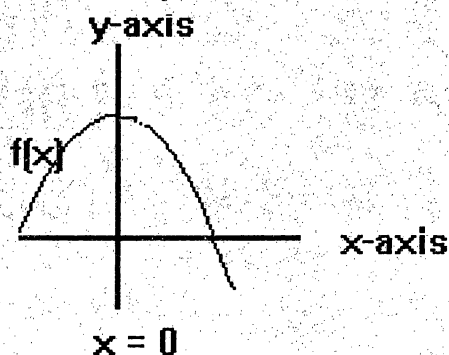
λ = wavelength

The speed of the wave is the distance the wave travels divided by the time traveled.

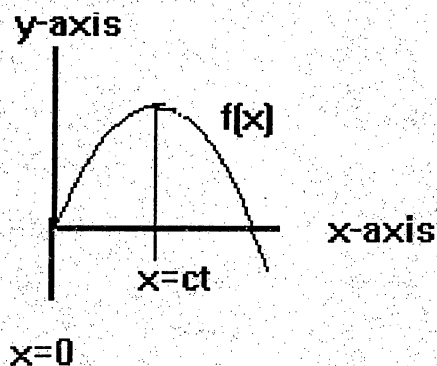
Therefore, the wave speed c , is $\frac{\text{distance}}{\text{time}} = \frac{\lambda}{P} = \frac{\lambda}{\frac{1}{f}}$

$$c = f\lambda$$

Suppose we have a traveling wave moving along the x -axis, and at $t = 0$, the peak of the wave, $f(x)$ is at $x = 0$. We can describe the pulse at this moment in time as the displacement y as a function of position x .

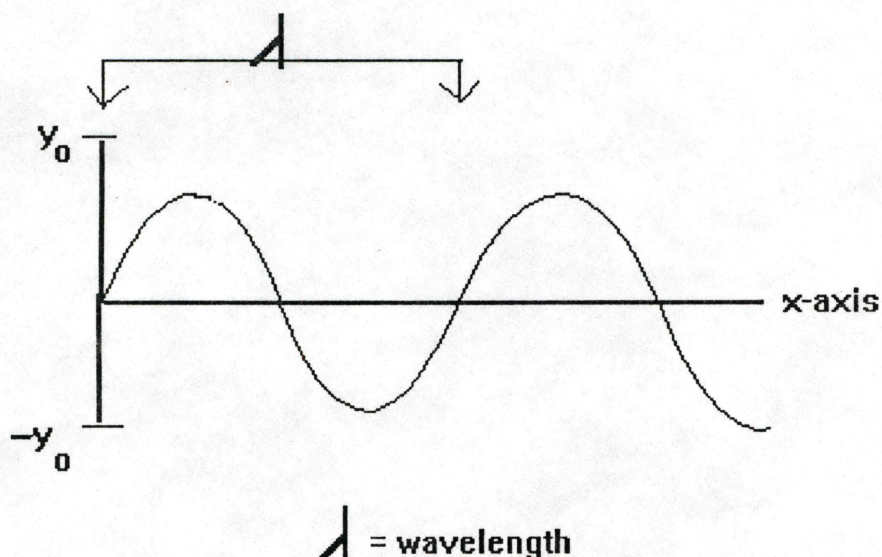


We need a function that gives the same shape as $f(x)$ but moved to the right. At a later arbitrary time t , the pulse of the wave can be pictured as the following;



The pulse is moving at speed c . Therefore it has moved a distance of ct . A function that represents this displacement with time t is $f(x - ct)$. So whatever the shape of the wave $f(x)$ describes, $f(x - ct)$ describes the same shape but at a later time t . The peak of the wave of $f(x)$ is at $x = 0$,

whereas the peak of the wave of $f(x - ct)$ is at $x = ct$. An example of such a wave at $t = 0$ is given by the equation $y = y_0 \cos(2\pi x/\lambda)$ where y is the vertical displacement from the point of equilibrium, the maximum displacement or amplitude is y_0 , and λ is the wavelength.



We need to describe the wave's position at time t . Replace x by $x - ct$.

$$y = y_0 \cos \frac{2\pi}{\lambda} (x - ct)$$

$$y = y_0 \cos(2\pi x/\lambda - 2\pi ct/\lambda)$$

$$c = f\lambda \quad \text{implies} \quad \frac{c}{\lambda} = f$$

$$y = y_0 \cos(2\pi x/\lambda - 2\pi ft)$$

Let $k = (2\pi/\lambda)$, where k equals the wave number. The wave number describes the number of radians of a wave cycle per unit distance.

Let $\omega = 2\pi f$.

Therefore we have $y = y_0 \cos(kx - \omega t)$ as a solution to our

equation,
$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{u} \frac{\partial^2 y}{\partial x^2}$$

Note: $y = y_0 \cos(kx + \omega t)$ is also a solution; it is a wave moving to the left along the x-axis. To relate the constant T/u in the wave equation to the constants in a periodic

solution we substitute the cosine solution into the wave equation:
$$\frac{\partial^2 y}{\partial t^2} = -y_0 \omega^2 \cos(kx - \omega t), \quad \frac{\partial^2 y}{\partial x^2} = -y_0 k^2 \cos(kx - \omega t)$$

then

$$\frac{T}{u} = \frac{\frac{\partial^2 y}{\partial t^2}}{\frac{\partial^2 y}{\partial x^2}} = \frac{-y_0 \omega^2 \cos(kx - \omega t)}{-y_0 k^2 \cos(kx - \omega t)}$$

$$\frac{T}{u} = \frac{\omega^2}{k^2}. \quad \text{But } \omega = 2\pi f \text{ and } k = \frac{2\pi}{\lambda}, \text{ so}$$

$$\frac{T}{u} = \frac{(2\pi f)^2}{(2\pi/\lambda)^2} = 4\pi^2 f^2 \frac{\lambda^2}{4\pi^2} = f^2 \lambda^2 = c^2.$$

So the equation for the string that vibrates near the point of equilibrium is $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$, where $c = \sqrt{\frac{T}{u}}$.

If $y = \Phi(x, t)$ is a periodic solution then we have shown that c is the wave speed. Recall that T is the tension in the string and u is the linear mass density. Observe that as the string becomes longer the wave speed slows, and as T grows the string is tightened and the speed of the wave is faster.

§1.2 SOLUTIONS TO THE WAVE EQUATION

We now want to look at what types of functions satisfy the wave equation. We will show equations of the form $y = f(x \pm ct)$ are solutions to the wave equation, where f is any function of one variable. [12] Since f is a function of one variable its derivatives can be defined as f' and f'' . Let f_1 and f_2 be arbitrary functions.

Show that $y = f_1(x - ct)$ is a solution to the wave equation,

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}.$$

Find the first and second partials of y with respect to t and x .

$$\frac{\partial y}{\partial t} = -cf_1'(x - ct).$$

$$\frac{\partial^2 y}{\partial t^2} = c^2 f_1''(x - ct)$$

and

$$\frac{\partial y}{\partial x} = f_1'(x - ct)$$

$$\frac{\partial^2 y}{\partial x^2} = f_1''(x - ct).$$

$$\text{So } \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}.$$

Therefore, $y = f_1(x - ct)$ is a solution to the wave equation.

Show that $y = f_2(x + ct)$ is a solution to the wave equation.

$$\frac{\partial y}{\partial t} = cf_2'(x + ct).$$

$$\frac{\partial^2 y}{\partial t^2} = c^2 f_2''(x + ct)$$

and

$$\frac{\partial y}{\partial x} = f_2'(x + ct)$$

$$\frac{\partial^2 y}{\partial x^2} = f_2''(x + ct).$$

Again, $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}.$

Therefore, $y = f_2(x + ct)$ is a solution to the wave equation. Because the wave equation is a linear equation, any linear combination $y = A f_1(x - ct) + B f_2(x + ct)$ is also a solution to the wave equation. Sine and cosine functions satisfy the wave equation, but functions such as exponentials will also satisfy the equation [2].

Show the wave equation has an exponential solution.

Let $y = e^{\alpha t} e^{\beta x}.$

Then

$$\frac{\partial y}{\partial t} = \alpha e^{\alpha t} e^{\beta x}$$

$$\frac{\partial^2 y}{\partial t^2} = \alpha^2 e^{\alpha t} e^{\beta x}$$

$$\frac{\partial y}{\partial x} = \beta e^{\alpha t} e^{\beta x}$$

$$\frac{\partial^2 y}{\partial x^2} = \beta^2 e^{\alpha t} e^{\beta x}.$$

Substitute the second partials into the wave equation.

$$\alpha^2 e^{\alpha t} e^{\beta x} = c^2 \beta^2 e^{\alpha t} e^{\beta x}$$

$$\alpha^2 = c^2 \beta^2$$

$$\beta = \frac{\pm \alpha}{c}.$$

$$\text{So } y = e^{\alpha t} e^{\pm \alpha x/c}$$

$$y_1 = e^{\alpha t + \alpha x/c} \quad \text{and}$$

$$y_2 = e^{\alpha t - \alpha x/c}$$

Therefore y_1 and y_2 are also solutions to the wave equation but not physically realistic. There are other solutions to the equation. To find them all we need Fourier Analysis, which is beyond the scope of this paper.

§1.3 SOLUTIONS OF THE FORM $\Phi = X(x)T(t)$

If we have a linear second order ordinary differential equation with constant coefficients, we can easily find the general solution. If the solution a partial differential equation can be represented as a product of independent functions $X(x)T(t)$, we can separate the functions and solve by using general solutions from O.D.E. Assume that $\Phi(x,t)$ is a solution of the form $\Phi = X(x)T(t)$, where X is a function of x , and T is a function of t .

$$\frac{\partial \Phi}{\partial x} = X'(x)T(t)$$

$$\frac{\partial \Phi}{\partial t} = X(x)T'(t)$$

$$\frac{\partial^2 \Phi}{\partial x^2} = X''(x)T(t)$$

$$\frac{\partial^2 \Phi}{\partial t^2} = X(x)T''(t)$$

Plug the second partials into the wave equation.

$$X''(x)T(t) = \frac{1}{c^2} X(x)T''(t)$$

Divide both sides by $\Phi = X(x)T(t)$.

$$\frac{X''(x)T(t)}{X(x)T(t)} = \frac{1}{c^2} \frac{X(x)T''(t)}{X(x)T(t)}$$

$$\frac{X''(x)}{X(x)} = \frac{1}{c^2} \frac{T''(t)}{T(t)}$$

The left hand side is a function of x only, and the right hand side is a function of t only. The only way this can happen is if both sides are equal to a constant. This constant is called the separation constant. Let the separation constant be a negative number. We use $-m^2$ instead of m^2 to insure periodic solutions, and so that later boundary conditions will be satisfied.

$$\frac{X''(x)}{X(x)} = \frac{1}{c^2} \frac{T''(t)}{T(t)} = -m^2$$

So $\frac{X''(x)}{X(x)} = -m^2$ and $\frac{1}{c^2} \frac{T''(t)}{T(t)} = -m^2$

$$X''(x) = -m^2 X(x) \quad \text{and} \quad T''(t) = -m^2 c^2 T(t)$$

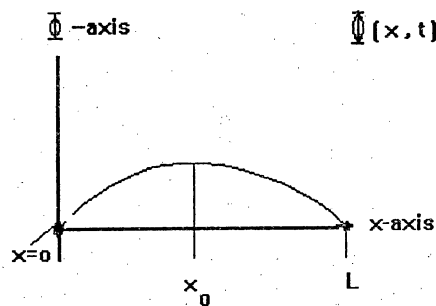
Both equations are second order ordinary differential equations. Two independent solutions to $X''(x) + m^2 X(x) = 0$ are $\Phi_1 = \sin(mx)$ and $\Phi_2 = \cos(mx)$. Two independent solutions to $T''(t) + m^2 c^2 T(t) = 0$ are $\Phi_3 = \sin(mct)$ and $\Phi_4 = \cos(mct)$. ([2], page 114). Hence, setting $X(x) = A \sin(mx) + B \cos(mx)$ and $T(t) = C \sin(mct) + D \cos(mct)$ a solution for the partial differential equation is

$$\Phi(x,t) = [A \sin(mx) + B \cos(mx)][C \sin(mct) + D \cos(mct)]$$

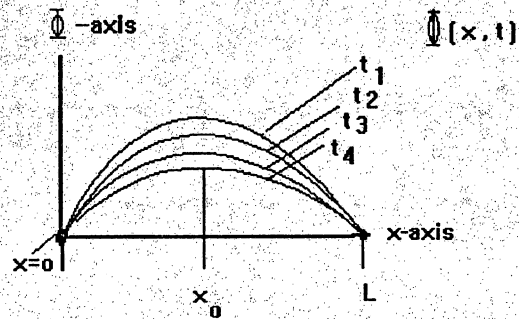
In chapter one we saw the derivation of the wave equation using physical properties. Solutions that satisfy the wave equation were calculated, and it was shown that arbitrary functions f_1, f_2 of the form $y = f_1(x - ct) + f_2(x + ct)$ satisfy the wave equation. In section three, it was demonstrated that $\Phi(x,t)$ has solutions of the form $\Phi = X(x)T(t)$.

§2.1 SOLVE THE WAVE EQUATION FOR THE STRING FIXED AT BOTH ENDS

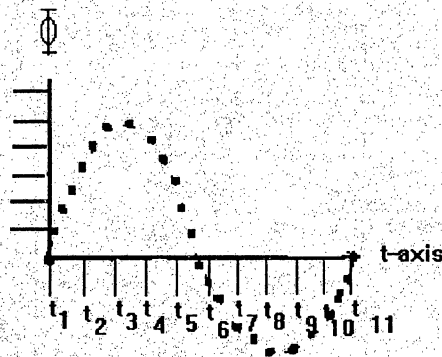
In chapter two, two cases for the vibrating string will be solved. The first case is where the string is fixed at both ends like that of a guitar or violin string. The second case is that of a string fixed at one end and the other end is allowed to travel up or down the line $x = x_0$. We need to formulate boundary conditions that describe the physical properties of the string. These boundary conditions will enable us to determine the constants A, B, C, D in the solution $\Phi(x, t) = [A \sin(mx) + B \cos(mx)][C \sin(mct) + D \cos(mct)]$.



This graph shows the position of the vibrating string at time t , with the ends fixed at $x=0$ and $x=L$, with vertical axis $y = \Phi$ and horizontal axis x .



This illustration shows the string fixed at both ends with different values of Φ at x_0 for different times t_i .



This illustration shows the string's position at x_0 for various times t_i .

Having the string fixed at both ends imposes boundary conditions on the wave equation.

The boundary conditions are as follows;

B.C #1 $\Phi(0,t)=0$ (This implies $X(0)=0 \forall t \in T$)

B.C.#2 $\Phi(L,t)=0$ (This implies $X(L)=0 \forall t \in T$)

B.C.#3 $\Phi(x,0)=0$ (This implies $T(0)=0 \forall x \in X$)

B.C.#4 $\frac{\partial \Phi(x_0, 0)}{\partial t} = v$ (For some $x_0 \in X$, $x_0 = v$, when $t=0$, where $v \in \mathbb{R}$ is the initial velocity of the point x_0)

A node is a place in which the function is always zero. Boundary conditions number one states the function has a node, at $x = 0$, which implies the string is at rest. Boundary condition number two states the length of the string is L , and at this point there is another node. Boundary condition number three states that at time zero the string is at rest, and boundary condition number four states the velocity of the string at a fixed point on the string at time zero is a constant v .

Assume that $\Phi(x, t)$ is a solution of the form

$$\Phi = X(x)T(t).$$

In section 1.3 we found a periodic solution

$$\Phi(x, t) = [A \sin(mx) + B \cos(mx)][D \sin(cmt) + E \cos(cmt)].$$

Test the boundary conditions.

B.C. #1 $\Phi(0, t) = 0$ implies

$$[A \sin(0) + B \cos(0)][D \sin(cmt) + E \cos(cmt)] = 0$$

$$B [D \sin(cmt) + E \cos(cmt)] = 0$$

$$\text{Either } B = 0 \text{ or } [D \sin(cmt) + E \cos(cmt)] = 0 \quad \forall t.$$

If $[D \sin(cmt) + E \cos(cmt)] = 0$ then $D \sin(cmt) = -E \cos(cmt)$, which is a contradiction since the sine and the cosine cannot be proportional, so $D = E = 0$. But this is the trivial wave, therefore in a nonzero solution, $B = 0$.

$$\text{So } \Phi(x, t) = A \sin(mx) [D \sin(cmt) + E \cos(cmt)].$$

B.C. #2 $\Phi(L, t) = 0$ implies

$$A \sin(mL) [D \sin(cmt) + E \cos(cmt)] = 0$$

Either $A \sin(mL) = 0$ or $D \sin(cmt) + E \cos(cmt) = 0 \forall t$.

If $D \sin(cmt) + E \cos(cmt) = 0$ then, we have the trivial wave again, therefore $A \sin(mL) = 0$.

$A \sin(mL) = 0$ implies $mL = k\pi$, $k \in \mathbb{Z}$

$$\text{Hence } m = \frac{k\pi}{L}.$$

So $\Phi(x, t) = A \sin(k\pi x/L) D \sin(ck\pi t/L) + E \cos(ck\pi t/L)$.

B.C. #3 $\Phi(x, 0) = 0$ implies,

$$A \sin(k\pi x/L) D \sin(0) + E \cos(0) = 0$$

$A \sin(k\pi x/L) E = 0$, therefore $E = 0$.

So $\Phi(x, t) = A \sin(k\pi x/L) D \sin(ck\pi t/L)$

Let $AD = G$,

$$\Phi(x, t) = G \sin(k\pi x/L) \sin(ck\pi t/L)$$

B.C. #4

$$\frac{\partial \Phi(x_0, 0)}{\partial t} = v$$

Find the derivative of $\Phi(x, t)$ at $(x_0, 0)$:

$$\frac{\partial \Phi}{\partial t} = (Gk\pi c/L) \sin(\pi k x_0/L) \cos(0) = v$$

$$(Gk\pi c/L) \sin(\pi k x_0/L) = v$$

$$G = \frac{vL}{k\pi c \sin(\pi k x_0)}$$

Therefore our solution under these boundary conditions is

$$\Phi(x, t) = \frac{vL \sin(\pi k x/L) \sin(\pi k c t/L)}{k\pi c \sin(\pi k x_0)}$$

Where v is velocity of the string at x_0 , L is the length of the string, c is the wave speed, and k is an arbitrary integer. We can rewrite the equation as

$$\frac{\partial^2}{\partial t^2} \left(\frac{vL \sin(\pi kx/L) \sin(\pi kct/L)}{k\pi c \sin(\pi kx_0)} \right)$$

$$= - \left(\frac{\pi kc}{L} \right)^2 \frac{vL \sin(\pi kx/L) \sin(\pi kct/L)}{k\pi c \sin(\pi kx_0)}.$$

This has the form of an operator times a function, which equals a constant times the same function. The solution Φ is an eigenfunction of the Laplacian $\Delta = \frac{\partial^2}{\partial x^2}$ with eigenvalue

$-(\pi k/L)^2$. The solution Φ is also an eigenfunction of the operator $\frac{\partial^2}{\partial t^2}$ with eigenvalue $-(\pi kc/L)^2$. The

eigenvalue is the negative of the square of the angular frequency ω , which is in radians per second. So $\omega = \pi kc/L$. The regular frequency f , is in cycles per second, and there are 2π radians per cycle. Therefore if we divide ω by 2π we get $f = kc/2L$. Hence the frequency that produces the sound that is emitted from a vibrating string fixed at both ends is $kc/2L$, where

$k = 1, 2, \dots$. As k runs through the integers an infinite number of frequencies are produced. What we hear is combinations of these frequencies. Certain combinations are perceived as noise, whereas other combinations are pleasant sounding and as music. So different frequencies produce different sounds, and the higher the frequency the higher the pitch. This implies

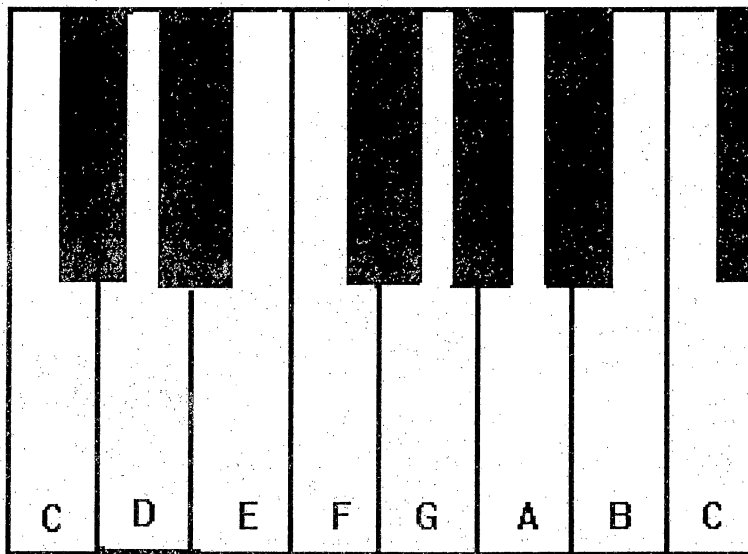
by our $kc/2L$, that the higher the pitch the faster the wave speed. Notice as L (the length of the string) gets larger, the pitch will be lower.

Sound that we perceive as musical notes have their frequencies in the following ratio form [13],

C	D	E	F	G	A	B	C'
1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2

For example the note E, has a frequency $5/4$ times that of the note C, and F has a frequency $4/3$ times that of C. The note C' has twice the frequency of C.

Consider one octave on a piano.



$$f_1 \quad \frac{9f_1}{8} \quad \frac{5f_1}{4} \quad \frac{4f_1}{3} \quad \frac{3f_1}{2} \quad \frac{5f_1}{3} \quad \frac{15f_1}{8} \quad 2f_1$$

There is no absolute standard of pitch. The definition of middle C, for example, has changed over the decades [13], but

the ratios defining a scale are determined by the wave equation, as we shall see. Currently it is agreed that A has a frequency of 440 hz. If the frequency of middle C = f_1 is taken to be 264 hertz, then G is 396 hz, and $C'=2f_1 = 528$ hz.

Also, as we go up an octave each of these ratios are in the following form

Up one octave from f_1 ,

$$D' = 2\left(\frac{9f_1}{8}\right) \quad E' = 2\left(\frac{5f_1}{4}\right) \quad F' = 2\left(\frac{4f_1}{3}\right)$$

$$D'' = 2^2\left(\frac{9f_1}{8}\right) \quad E'' = 2^2\left(\frac{5f_1}{4}\right) \quad F'' = 2^2\left(\frac{4f_1}{3}\right)$$

$$D''' = 2^3\left(\frac{9f_1}{8}\right) \quad E''' = 2^3\left(\frac{5f_1}{4}\right) \quad F''' = 2^3\left(\frac{4f_1}{3}\right)$$

Consider the frequencies that appear in the solution of the wave equation. The corresponding notes are

$$C_1 = 1f_1 = (\text{the fundamental frequency})$$

$$C_2 = 2f_1 = (\text{the first harmonic or overtone})$$

$$G_2 = 2(3/2)f_1 = 3f_1 \quad (\text{the second harmonic or overtone})$$

$$C_3 = 2^2f_1 = 4f_1$$

$$E_3 = 2^2(5/4)f_1 = 5f_1$$

$$G_3 = 2^2(3/2)f_1 = 6f_1$$

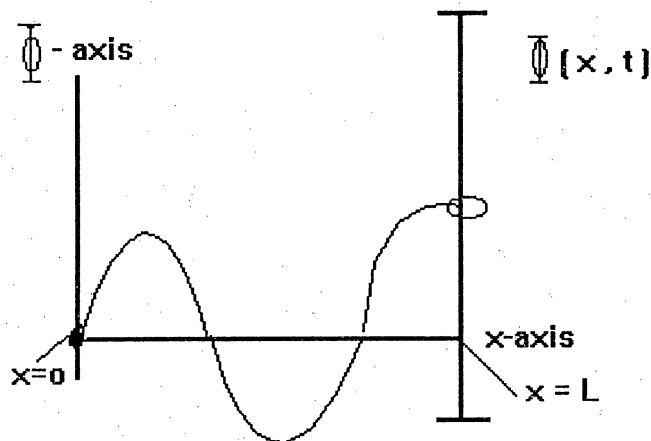
$$B_3^b = 2^2(7/4)f_1 = 7f_1$$

$$C_4 = 2^3f_1 = 8f_1$$

Recall $f = kc/2L$ where $k = 1, 2, 3, \dots$. This is what just came out of our calculations. When a guitar string is plucked, the sound is a combination of these notes. The fundamental tone is the most prominent.

§2.2 SOLVE THE WAVE EQUATION FOR A STRING FIXED AT ONE END

We will now solve the wave equation for the string fixed at one end, and allowed to move up and down the line $\Phi(x) = x_0$.



Boundary Conditions,

B.C. #1 $\Phi(0,t) = 0$ (This implies $X(0)=0 \forall t \in T$)

B.C. #2 $\frac{\partial \Phi}{\partial x}(L,t)=0$ (This implies that $\forall t \in T$ the slope of the string is zero at $x = L$)

B.C. #3 $\Phi(x,0)=0$ (This implies that function is zero at $t=0$)

B.C. #4 $\frac{\partial \Phi}{\partial t}(x_0,0)=v$ (This implies that the piece of the string at x_0 has velocity v at $t=0$)

Assume the solution to be of the form $\Phi(x,t)=X(x)T(t)$.

From previous work in section 1.3, a solution is

$$\Phi(x,t)=[A \sin(mx) + B \cos(mx)][C \sin(mct) + D \cos(mct)].$$

Test the boundary conditions.

B.C. #1 $\phi(0,t) = 0$ implies

$$[A \sin(0) + B \cos(0)][C \sin(mct) + D \cos(mct)] = 0$$

so either $[A \sin(0) + B \cos(0)] = 0$ or

$$[C \sin(mct) + D \cos(mct)] = 0, \text{ contradiction unless}$$

$C = D = 0$ which implies the trivial solution. The trivial solution is a flat wave which is not very interesting, therefore $B = 0$. Our solution is

$$\phi(x,t) = A \sin(mx) [C \sin(mct) + D \cos(mct)]$$

B.C. #2 $\frac{\partial \phi}{\partial x}(L,t) = 0$ implies

$$\frac{\partial \phi}{\partial x} = Am \sin(mL) (C \sin(mct) + D \cos(mct)) = 0,$$

So either $Am \sin(mL) = 0$ or $C \sin(mct) + D \cos(mct) = 0$, as before contradiction unless $C = D = 0$, which is the trivial wave, so $Am \sin(mL) = 0$. Therefore $m = \pi k/L$ where $k \in \mathbb{Z}$.

$$\phi(x,t) = A\pi k/L \sin(\pi kx/L) (C \sin(\pi kct/L) + D \cos(\pi kct/L))$$

B.C. #3 $\phi(x,0) = 0$ implies

$A\pi k/L \sin(\pi kx/L) (C \sin(0) + D \cos(0)) = 0$. So $D = 0$, and our solution is

$$\phi(x,t) = A\pi k/L \sin(\pi kx/L) C \sin(\pi kct/L). \text{ Let } AC\pi k/L = F;$$

$$\text{then } \phi(x,t) = F \sin(\pi kx/L) \sin(\pi kct/L)$$

B.C. #4 $\frac{\partial \phi}{\partial t}(x_0,0) = v$ implies

$$\frac{\partial \phi}{\partial t}(x_0,0) = F \sin(\pi kx_0/L) \pi kc/L \cos(0) = v$$

$$F \sin(\pi kx_0/L) \pi kc/L = v \text{ which implies}$$

$$F = vL/(\pi kc \sin(\pi kx_0/L))$$

So our solution for a string fixed at one end is

$$\Phi(x,t) = \frac{vL}{\pi kc} \frac{\sin(\pi kx/L) \sin(\pi kct/L)}{\sin(\pi kx_0/L)}$$

As before our eigenvalue is $-(\pi kc/L)^2$, and our frequency is $kc/2L$.

In chapter two we have seen the solutions for the string fixed at both ends, and for the string fixed at one end. From the particular solution an eigenvalue was found which was the negative of the square of the angular frequency. From the angular frequency the regular frequency was calculated by dividing by 2π . The frequency could then be correlated to notes on a musical scale which we perceive as pleasant sound.

§3.1 THE WAVE EQUATION IN R^3 IN CYLINDRICAL COORDINATES

In the previous sections we have worked with the wave equation involving just x and t . To solve the wave equation on the cylinder, we need the wave equation in three dimensions, $\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2}$. Since we will be solving the wave equation for the cylinder, it will be easier to use cylindrical coordinates. We will now change the three dimensional wave equation from rectangular coordinates into cylindrical coordinates. We will then solve the equation for the cylinder. Change (x, y, z) to (r, θ, z) , where $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$ [4]. From the above information we can derive the following partials.

$$\frac{\partial r}{\partial x} = \cos \theta \quad \frac{\partial \theta}{\partial x} = -\frac{\sin \theta}{r}$$

We will assume that Φ is C^2 ; that is all second partial exist and are continuous. Therefore mixed partials are equal.

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial \Phi}{\partial \theta} \frac{\partial \theta}{\partial x}$$

$$\frac{\partial \Phi}{\partial y} = \frac{\partial \Phi}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial \Phi}{\partial \theta} \frac{\partial \theta}{\partial y}$$

$$\frac{\partial \Phi}{\partial z} = \frac{\partial \Phi}{\partial z}$$

We want the second partial of Φ :

$$\frac{\partial^2 \Phi}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial \Phi}{\partial r} \frac{\partial r}{\partial x} \right) + \frac{\partial}{\partial x} \left(\frac{\partial \Phi}{\partial \theta} \frac{\partial \theta}{\partial x} \right)$$

$$\begin{aligned}
&= \frac{\partial}{\partial x} \left(\frac{\partial \Phi}{\partial r} \right) \frac{\partial r}{\partial x} + \frac{\partial}{\partial x} \left(\frac{\partial r}{\partial x} \right) \frac{\partial \Phi}{\partial r} + \frac{\partial}{\partial x} \left(\frac{\partial \Phi}{\partial \theta} \right) \frac{\partial \theta}{\partial x} + \frac{\partial}{\partial x} \left(\frac{\partial \theta}{\partial x} \right) \frac{\partial \Phi}{\partial \theta} \\
&= \frac{\partial f_1}{\partial x} \cos \theta + \frac{\partial}{\partial x} (\cos \theta) \frac{\partial \Phi}{\partial r} + \frac{\partial f_2}{\partial x} \left(-\frac{1}{r} \sin \theta \right) + \frac{\partial}{\partial x} \left(-\frac{1}{r} \sin \theta \right) \frac{\partial \Phi}{\partial \theta}
\end{aligned}$$

where $f_1 = \frac{\partial \Phi}{\partial r}$ and $f_2 = \frac{\partial \Phi}{\partial \theta}$.

$$= \frac{\partial f_1}{\partial x} \cos \theta + \frac{\partial f_2}{\partial x} \left(-\frac{1}{r} \sin \theta \right) \quad (\text{equation 1})$$

$$\frac{\partial f_1}{\partial x} = \frac{\partial}{\partial r} \left(\frac{\partial \Phi}{\partial r} \right) \frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta} \left(\frac{\partial \Phi}{\partial r} \right) \frac{\partial \theta}{\partial x}$$

$$\frac{\partial f_1}{\partial x} = \frac{\partial^2 \Phi}{\partial r^2} \cos \theta - \frac{1}{r} \sin \theta \frac{\partial^2 \Phi}{\partial \theta \partial r}$$

$$\frac{\partial f_2}{\partial x} = \frac{\partial}{\partial r} \left(\frac{\partial \Phi}{\partial \theta} \right) \frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta} \left(\frac{\partial \Phi}{\partial \theta} \right) \frac{\partial \theta}{\partial x}$$

$$\frac{\partial f_2}{\partial x} = \frac{\partial^2 \Phi}{\partial r \partial \theta} \cos \theta - \frac{1}{r} \sin \theta \frac{\partial^2 \Phi}{\partial \theta^2}$$

Using equation 1, substitute $\frac{\partial f_1}{\partial x}$ and $\frac{\partial f_2}{\partial x}$.

$$\begin{aligned}
\frac{\partial^2 \Phi}{\partial x^2} &= \cos \theta \left(\cos \theta \frac{\partial^2 \Phi}{\partial r^2} - \frac{1}{r} \sin \theta \frac{\partial^2 \Phi}{\partial \theta \partial r} \right) \\
&\quad - \frac{1}{r} \sin \theta \left(\frac{\partial^2 \Phi}{\partial r \partial \theta} \cos \theta - \frac{1}{r} \sin \theta \frac{\partial^2 \Phi}{\partial \theta^2} \right) \\
&= \cos \theta \frac{\partial}{\partial r} \left(\cos \theta \frac{\partial \Phi}{\partial r} - \frac{1}{r} \sin \theta \frac{\partial \Phi}{\partial \theta} \right) \\
&\quad - \frac{1}{r} \sin \theta \frac{\partial}{\partial \theta} \left(\cos \theta \frac{\partial \Phi}{\partial r} - \frac{1}{r} \sin \theta \frac{\partial \Phi}{\partial \theta} \right)
\end{aligned}$$

$$\begin{aligned}
&= \cos\theta \left(\frac{\partial^2 \Phi}{\partial r^2} \cos\theta + \frac{1}{r^2} \sin\theta \frac{\partial \Phi}{\partial \theta} - \frac{1}{r} \sin\theta \frac{\partial \Phi}{\partial r \partial \theta} \right) \\
&- \frac{1}{r} \sin\theta \left(\frac{\partial^2 \Phi}{\partial \theta \partial r} \cos\theta - \sin\theta \frac{\partial \Phi}{\partial r} - \frac{1}{r} \cos\theta \frac{\partial \Phi}{\partial \theta} - \frac{1}{r} \sin\theta \frac{\partial \Phi}{\partial \theta} \right) \\
&= \cos^2\theta \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r^2} \sin\theta \cos\theta \frac{\partial \Phi}{\partial \theta} - \frac{1}{r} \sin\theta \cos\theta \frac{\partial^2 \Phi}{\partial r \partial \theta} \\
&- \frac{1}{r} \sin\theta \cos\theta \frac{\partial^2 \Phi}{\partial \theta \partial r} + \frac{1}{r} \sin^2\theta \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \sin\theta \cos\theta \frac{\partial \Phi}{\partial \theta} \\
&+ \frac{1}{r^2} \sin^2\theta \frac{\partial \Phi}{\partial \theta}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{\partial^2 \Phi}{\partial x^2} &= \cos^2\theta \frac{\partial^2 \Phi}{\partial r^2} + \frac{2}{r^2} \sin\theta \cos\theta \frac{\partial \Phi}{\partial \theta} - \frac{2}{r} \sin\theta \cos\theta \frac{\partial^2 \Phi}{\partial r \partial \theta} \\
&+ \frac{1}{r} \sin^2\theta \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \sin^2\theta \frac{\partial^2 \Phi}{\partial \theta^2}.
\end{aligned}$$

A similar derivation shows that

$$\begin{aligned}
\frac{\partial^2 \Phi}{\partial y^2} &= \sin^2\theta \frac{\partial^2 \Phi}{\partial r^2} + \frac{2}{r} \sin\theta \cos\theta \frac{\partial^2 \Phi}{\partial \theta \partial r} \\
&- \frac{2}{r^2} \sin\theta \cos\theta \frac{\partial \Phi}{\partial \theta} + \frac{1}{r} \cos^2\theta \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \cos^2\theta \frac{\partial^2 \Phi}{\partial \theta^2}.
\end{aligned}$$

Plug the partial derivatives into the wave equation

$$\begin{aligned}
\frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} &= \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \\
\frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} &= \cos^2\theta \frac{\partial^2 \Phi}{\partial r^2} + \frac{2}{r^2} \sin\theta \cos\theta \frac{\partial \Phi}{\partial \theta} \\
&- \frac{2}{r} \sin\theta \cos\theta \frac{\partial^2 \Phi}{\partial r \partial \theta} + \frac{1}{r} \sin^2\theta \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \sin^2\theta \frac{\partial^2 \Phi}{\partial \theta^2}
\end{aligned}$$

$$+ \sin^2\theta \frac{\partial^2\Phi}{\partial r^2} + \frac{2}{r} \sin\theta\cos\theta \frac{\partial^2\Phi}{\partial\theta\partial r} - \frac{2}{r^2} \sin\theta\cos\theta \frac{\partial\Phi}{\partial\theta}$$

$$+ \frac{1}{r} \cos^2\theta \frac{\partial\Phi}{\partial r} + \frac{1}{r^2} \cos^2\theta \frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2}$$

$$\frac{1}{c^2\partial t^2} = \cos^2\theta \frac{\partial^2\Phi}{\partial r^2} + \sin^2\theta \frac{\partial^2\Phi}{\partial r^2}$$

$$+ \frac{1}{r^2} \sin^2\theta \frac{\partial^2\Phi}{\partial\theta^2} + \frac{1}{r^2} \cos^2\theta \frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2}$$

$$\frac{1}{c^2\partial t^2} = \frac{\partial^2\Phi}{\partial r^2} (\cos^2\theta + \sin^2\theta) + \frac{1}{r^2} \frac{\partial^2\Phi}{\partial\theta^2} (\cos^2\theta + \sin^2\theta)$$

$$+ \frac{1}{r} \frac{\partial\Phi}{\partial r} (\cos^2\theta + \sin^2\theta) + \frac{\partial^2\Phi}{\partial z^2}$$

$$\frac{1}{c^2\partial t^2} = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2\Phi}{\partial\theta^2} + \frac{1}{r} \frac{\partial\Phi}{\partial r} + \frac{\partial^2\Phi}{\partial z^2}$$

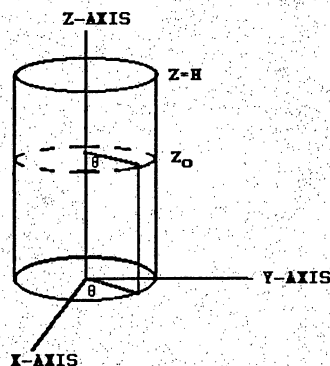
$$\text{Note: } \frac{1}{r} \left[\frac{\partial}{\partial r} \left(r \frac{\partial\Phi}{\partial r} \right) \right] = \frac{1}{r} \left[\frac{\partial\Phi}{\partial r} + r \frac{\partial^2\Phi}{\partial r^2} \right] = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r} \frac{\partial\Phi}{\partial r}.$$

Therefore the wave equation in cylindrical coordinates is

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial\Phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2} = \frac{1}{c^2\partial t^2}$$

§3.2 SOLVE THE WAVE EQUATION FOR THE CYLINDER

The case of the cylinder can be generalized from the case of the vibrating string. The string oscillates up and down in a plane until eventually coming to rest at the point of equilibrium. The restoring forces pull the string toward the point of equilibrium. Likewise, the cylinder vibrates radially, as the restoring forces pull the wave back toward the point of equilibrium which is the surface of the cylinder. The direction of vibration can be thought of as a normal vector to the surface of the cylinder. We will assume that both ends of the cylinder are clamped, which is analogous to the string fixed at both ends.



In cylindrical coordinates the wave equation is

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2}$$

The radius of the cylinder is a constant R and so all derivatives with respect to r are zero; therefore, the first term is zero. So our wave equation becomes,

$$\frac{1}{R^2} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2}, \quad \text{where } \Phi = \Phi(\theta, z, t) \text{ a function}$$

of three variables.

Boundary conditions:

- 1) $\Phi(\theta, z, 0) = 0$ (Implies the cylinder is initially at rest
 $\forall \theta, z$)
- 2) $\Phi(\theta, 0, t) = 0$ (Implies a node at the bottom of the cylinder
 $\forall \theta, t$)
- 3) $\Phi(\theta, H, t) = 0$ (Implies a node at the top of the cylinder
 $\forall \theta, t$)
- 4) $\frac{\partial \Phi}{\partial t}(\theta_0, z_0, 0) = v$

Boundary condition number one implies at time equal zero, the system is at rest. Conditions number two and three state that there is a node at every point along the bottom of the cylinder, and at every point along the top of the cylinder. Recall the nodes are places on the wave where the function is identically zero. Condition number four implies that at time zero, the velocity at a specific point on the cylinder θ_0, z_0 , is v .

Assume $\Phi = \Phi(\theta, z, t) = \theta(\theta)Z(z)T(t)$

$$\frac{\partial^2 \Phi}{\partial \theta^2} = \theta''(\theta)Z(z)T(t)$$

$$\frac{\partial^2 \Phi}{\partial z^2} = \theta(\theta)Z''(z)T(t)$$

$$\frac{\partial^2 \Phi}{\partial t^2} = \theta(\theta)Z(z)T''(t)$$

Substitute the partial derivatives into wave equation

$$\frac{1}{R^2} \theta''(\theta)Z(z)T(t) + \theta(\theta)Z''(z)T(t) = \frac{1}{c^2} \theta(\theta)Z(z)T''(t)$$

$$\frac{1}{R^2} \theta''(\theta) Z(z) T(t) + \theta(\theta) Z'(z) = \frac{1}{c^2} \theta(\theta) Z(z) \frac{T''(t)}{T(t)}$$

$$\frac{1}{R^2} \frac{\theta''(\theta)}{\theta(\theta)} + \frac{Z''(z)}{Z(z)} = \frac{1}{c^2} \frac{T''(t)}{T(t)}$$

The left hand side is a function of θ and Z , and the right hand side is a function of T . Both sides are independent, so they must equal a constant. Let the constant equal $-m^2$.

$$\frac{1}{c^2} \frac{T''(t)}{T(t)} = -m^2 \quad \text{implies} \quad T''(t) + m^2 c^2 T(t) = 0$$

$$T(t) = A \sin(mct) + B \cos(mct)$$

Test boundary conditions #1.

Notice this is work from the one dimensional case.

$$\text{B.C. \#1} \quad \Phi(\theta, z, 0) = 0$$

$$T(0) \theta(\theta) Z(z) = 0$$

$$T(0) = A \sin(0) + B \cos(0) \quad \text{which implies} \quad B=0.$$

$$\text{so } T(t) = A \sin(mct)$$

Separate the left hand side.

$$\frac{1}{R^2} \frac{\theta''(\theta)}{\theta(\theta)} + \frac{Z''(z)}{Z(z)} = -m^2$$

$$\frac{1}{R^2} \frac{\theta''(\theta)}{\theta(\theta)} = -m^2 - \frac{Z''(z)}{Z(z)}$$

Again, both sides must be constant. Let the constant be $-n^2$.

$$\frac{1}{R^2} \frac{\theta''(\theta)}{\theta(\theta)} = -n^2 \quad \text{implies} \quad \theta''(\theta) + n^2 R^2 \theta(\theta) = 0$$

$$\text{So } \theta(\theta) = C \sin(nR\theta) + D \cos(nR\theta)$$

$$\frac{Z''(z)}{Z(z)} + m^2 = -n^2 \quad \text{implies}$$

$$Z''(z) + (m^2 - n^2) Z(z) = 0 \quad \text{has a solution of the form}$$

$$Z(z) = E \sin(\sqrt{m^2 - n^2} z) + F \cos(\sqrt{m^2 - n^2} z)$$

$$\text{B.C. \#2 } \Phi(\theta, 0, t) = 0$$

$$Z(0) = E \sin(0) + F \cos(0) = 0 \text{ implies } F = 0$$

$$Z(z) = E \sin(\sqrt{m^2 - n^2} z)$$

$$\text{B.C. \# 3 } \Phi(\theta, H, t) = 0$$

$$Z(H) = E \sin(\sqrt{m^2 - n^2} H) = 0$$

$$\text{Let } (\sqrt{m^2 - n^2})H = p\pi \quad \ni p \in \mathbb{Z}$$

$$\sqrt{m^2 - n^2} = \frac{p\pi}{H}$$

$$m = (p^2\pi^2 - n^2H^2)^{1/2} / H$$

$$\text{So } Z(z) = E \sin(p\pi z/H)$$

$$\text{Now } T(t) = A \sin(ct[p^2\pi^2 - n^2H^2]^{1/2}/H)$$

$$\theta(\theta) = C \sin(nR\theta) + D \cos(nR\theta)$$

$$Z(z) = E \sin(p\pi z/H)$$

$$\Phi = (C \sin(nR\theta) + D \cos(nR\theta))$$

$$\cdot E \sin(p\pi z/H) \cdot A \sin(ct[p^2\pi^2 - n^2H^2]^{1/2}/H)$$

$$\text{For solutions to be defined on the cylinder, } nR = j$$

$$\text{where } j \in \mathbb{Z}. \text{ So } n = j/R$$

$$\text{Let } G = EA.$$

$$\Phi = G(C \sin(j\theta) + D \cos(j\theta))$$

$$\cdot \sin(p\pi z/H) \sin(ct[p^2\pi^2 - j^2H^2/R^2]^{1/2}/H)$$

$$\text{B.C. \#4 } \frac{\partial \Phi}{\partial t}(\theta_0, z_0, 0) = v$$

$$\frac{\partial \Phi}{\partial t} = G(C \sin(j\theta_0) + D \cos(j\theta_0))$$

$$\cdot \sin(p\pi z_0/H) \cdot \cos(0) \cdot c(p^2\pi^2 - j^2H^2/R^2)^{1/2}/H = v$$

$$v = Gc/H (C \sin(j\theta_0) + D \cos(j\theta_0))$$

$$\cdot \sin(p\pi z_0/H) \cdot (p^2\pi^2 - j^2H^2/R^2)^{1/2}$$

$$G = \frac{v}{a b c d}$$

$$a = (p^2 \pi^2 - j^2 H^2 / R^2)^{1/2} / H$$

$$b = \sin(p\pi z_0 / H)$$

$$d = C \sin(j\theta_0) + D \cos(j\theta_0).$$

Therefore the solution, subject to these boundary conditions is

$$\Phi = \frac{v [C \sin(j\theta) + D \cos(j\theta)] \sin(p\pi z / H) \sin(act)}{a b c d}.$$

Using a trig identity, let

$$C \sin(j\theta) + D \cos(j\theta) = \sqrt{C^2 + D^2} \sin(j\theta - \xi) \text{ where } \tan \xi = D/C,$$

and

$$C \sin(j\theta_0) + D \cos(j\theta_0) = \sqrt{C^2 + D^2} \sin(j\theta_0 - \xi).$$

Therefore

$$\Phi = \frac{v (\sqrt{C^2 + D^2}) \sin(j\theta - \xi) \sin(p\pi z / H) \sin(act)}{a c b (\sqrt{C^2 + D^2}) \sin(j\theta_0 - \xi)}$$

$$\Phi = \frac{v \sin(j\theta - \xi) \sin(p\pi z / H) \sin(act)}{c b \sin(j\theta_0 - \xi)}.$$

Therefore replacing a and b our solution becomes

$$\Phi = \frac{v H \sin(j\theta - \xi) \sin(p\pi z / H) \sin(ct(p^2 \pi^2 - j^2 H^2 / R^2)^{1/2} / H)}{c \sin(p\pi z_0 / H) \sin(j\theta_0 - \xi) (p^2 \pi^2 - j^2 H^2 / R^2)^{1/2}}.$$

Where R and H are the radius and the height of the cylinder respectively, c is the wave speed, v is the initial velocity of a point on the cylinder of the wave, and p and j are

integers. The solution Φ is an eigenfunction of the Laplacian and also an eigenfunction of $\frac{\partial^2}{\partial t^2}$ with eigenvalue

$-\frac{c^2}{H^2}(p^2\pi^2 - j^2/R^2)$. The frequency can be calculated as before.

It can be seen from the particular solution, that the radius and height of the cylinder have an effect on the sound. If the cylinder is shortened the pitch will become higher, and if the radius is widened the pitch will become lower.

In conclusion, a connection between sound and mathematics is the eigenvalue of a particular solution for a second order partial differential equation. Once the eigenvalue is known, the angular frequency and then the regular frequency can be calculated for the wave. The frequency then corresponds to a note from the musical scale, and we perceive these frequencies as music. If we were to observe the frequency on an oscilloscope, which breaks down the frequency by components, we would see the different waves which makes up the harmonics.

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